Dissipative Mechanics Using Complex-Valued Hamiltonians

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February 1, 2008

Abstract

We show that a large class of dissipative systems can be brought to a canonical form by introducing complex co-ordinates in phase space and a complex-valued hamiltonian. A naive canonical quantization of these systems lead to non-hermitean hamiltonian operators. The excited states are unstable and decay to the ground state . We also compute the tunneling amplitude across a potential barrier.

1 Introduction

In many physical situations, loss of energy of the system under study to the outside environment cannot be ignored. Often, the long time behavior of the system is determined by this loss of energy, leading to interesting phenomena such as attractors.

There is an extensive literature on dissipative systems at both the classical and quantum levels (See for example the textbooks [1, 2, 3]). Often the theory is based on an evolution equation of the density matrix of a 'small system' coupled to a 'reservoir' with a large number of degrees of freedom, after the reservoir has been averaged out. In such approaches the system is described by a mixed state rather than a pure state: in quantum mechanics by a density instead of a wavefunction and in classical mechanics by a density function rather than a point in the phase space.

There are other approaches that do deal with the evolution equations of a pure state. The canonical formulation of classical mechanics does not apply in a direct way to dissipative systems because the hamiltonian usually has the meaning of energy and would be conserved. By redefining the Poisson brackets [4] , or by using time dependent hamiltonians [5], it is possible to bring such systems within a canonical framework. Also, there are generalizations of the Poisson bracket that may not be anti-symmetric and/or may not satisfy the Jacobi identity [6, 7] which give dissipative equations.

We will follow another route, which turns out in many cases to be simpler than the above. It is suggested by the simplest example, that of the damped simple harmonic oscillator. As is well known, the effect of damping is to replace the natural frequency of oscillation by a complex number, the imaginary part of which

determines the rate of exponential decay of energy. Any initial state will decay to the ground state (of zero energy) as time tends to infinity. The corresponding coordinates in phase space (normal modes) are complex as well. This suggests that the equations are of hamiltonian form, but with a complex-valued hamiltonian.

It is not difficult to verify that this is true directly. The real part of the hamiltonian is a harmonic oscillator, although with a shifted frequency; the imaginary part is its constant multiple. If we pass to the quantum theory in the usual way, we get a non-hermitean hamiltonian operator. Its eigenvalues are complex valued, except for the ground state which can be chosen to have a real eigenvalue. Thus all states except the ground state are unstable. Any state decays to its projection to the ground state as time tends to infinity. This is a reasonable quantum analogue of the classical decay of energy.

We will show that a wide class of dissipative systems can be brought to such a canonical form using a complex-valued hamiltonian. The usual equations of motion determined by a hamiltonian and Poisson bracket are

$$
\frac{d}{dt} \begin{pmatrix} p \\ x \end{pmatrix} = \begin{pmatrix} \{H, p\} \\ \{H, x\} \end{pmatrix}.
\tag{1}
$$

At first a complex-valued function

$$
\mathcal{H} = H_1 + iH_2
\tag{2}
$$

does not seem to make sense when put into the above formula:

$$
\frac{d}{dt} \begin{pmatrix} p \\ x \end{pmatrix} = \begin{pmatrix} \{H_1, p\} \\ \{H_1, x\} \end{pmatrix} + i \begin{pmatrix} \{H_2, p\} \\ \{H_2, x\} \end{pmatrix}
\tag{3}
$$

since the l.h.s. has real components. How can we make sense of multiplication by i and still get a vector with only real components?

Let us consider a complex number $z = x + iy$ as an ordered pair of real numbers (x, y). The effect of multiplying z by i is the linear transformation

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} -y \\ x \end{pmatrix} \tag{4}$$

on its components. That is, multiplication by i is equivalent to the action by the matrix

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{5}$$

Note that $J^2 = -1$. Geometrically, this corresponds to a rotation by ninety degrees.

Generalizing this, we can interpret multiplication by i of a vector field in phase space to mean the action by some matrix J satisfying

$$J^2 = -1. \tag{6}$$

Given such a matrix, we can define the equations of motion generated by a complex-valued function $\mathcal{H} = H_1 + iH_2$ to be

$$\frac{d}{dt} \begin{pmatrix} p \\ x \end{pmatrix} = \begin{pmatrix} \{H_1, p\} \\ \{H_1, x\} \end{pmatrix} + J \begin{pmatrix} \{H_2, p\} \\ \{H_2, x\} \end{pmatrix} \tag{7}$$

Our point is that the infinitesimal time evolution of a wide class of mechanical systems is of this type for an appropriate choice of $\{,\}$, J, H_1 and H_2.

In most cases there is a complex-cordinate system in which J reduces to a simple multiplication by i; for example on the plane this is just $z = x + iy$. For such a co-ordinate system to exist the tensor field has to satisfy certain integrability conditions in addition to (6) above. These conditions are automatically satisfied if the matrix elements of J are constants.

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What would be the advantage of fitting dissipative systems into such a complex canonical formalism? A practical advantage is that they can lead to better numerical approximations, generalizing the symplectic integrators widely used in hamiltonian systems: these integrators preserve the geometric structure of the underlying physical system. Another is that it allows us to use ideas from hamiltonian mechanics to study structures unique to dissipative systems such as strange attractors. We will not pursue these ideas in this paper.

Instead we will look into the canonical quantization of dissipative systems. The usual correspondence principle leads to a non-hermitean hamiltonian. As in the elementary example of the damped simple harmonic oscillator, the eigenvalues are complex-valued. The excited states are unstable and decay to the ground state. Non-hermitean hamiltonians have arised already in several dissipative systems in condensed matter physics [9] and in particle physics [10]. The Wigner-Weisskopf approximation provides a physical justification for using a non-hermitean hamiltonian. A dissipative system is modelled by coupling it to some other 'external' degrees of freedom so that the total hamiltonian is hermitean and is conserved. In second order perturbation theory we can eliminate the external degrees of freedom to get an effective hamiltonian that is non-hermitean.

It is interesting to compare our approach with the tradition of Caldeira and Leggett[8]. Dissipation is modelled by coupling the original ('small') system to a thermal bath of harmonic oscillators. After integrating out the oscillators in the path integral formalism an effective action for the small system is obtained. A complication is that this effective action is non-local: its extremum (which dominates tunneling) is the solution of an integro-differential equation. We will see that the integral operator appearing here is also a complex structure (the Hilbert

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transform), although one non-local in time and hence different from our use of complex structures.

We calculate the tunneling amplitude of a simple one dimensional quantum system within our framework. Dissipation can increase the tunneling probability, which is not allowed in the Caldeira-Leggett model.

We begin with a brief review of the most elementary case, the damped simple harmonic oscillator. Then we generalize to the case of a generic one dimensional system with a dissipative force proportional to velocity. Further generalization to systems with several degrees of freedom is shown to be possible provided that the dissipative force is of the form

$$- \partial_a \partial_b W \frac{dx^b}{dt} \tag{8}$$

for some function W. In simple cases this function is just the square of the distance from the stable equilibrium point. Finally, we show how to bring a dissipative system whose configuration space is a Riemannian manifold into this framework. This is important to include interesting systems such as the rigid body or a particle moving on a curved surface. We hope to return to these examples in a later paper.

2   Dissipative Simple Harmonic Oscillator

We start by recalling the most elementary example of a classical dissipative system, described the differential equation

$$\ddot{x} + 2\gamma\dot{x} + \omega^2 x = 0, \gamma > 0. \tag{9}$$

We will consider the under-damped case $\gamma < \omega$ so that the system is still oscillatory.

We can write these equations in phase space

$$\dot{x} \;=\; p \tag{10}$$

$$\dot{p} \;=\; -2\gamma p - \omega^2 x \tag{11}$$

The energy

$$H = \tfrac{1}{2}[p^2 + \omega^2 x^2] \tag{12}$$

decreases monotonically along the trajectory:

$$\frac{dH}{dt} = p\dot{p} + \omega^2 x\dot{x} = -2\gamma p^2 \leq 0. \tag{13}$$

The only trajectory which conserves energy is the one with $p = 0$, which must have $x = 0$ as well to satisfy the equations of motion.

These equations can be brought to diagonal form by a linear transformation:

$$z = A\left[-i(p + \gamma x) + \omega_1 x\right], \quad \frac{dz}{dt} = [-\gamma + i\omega_1]z \tag{14}$$

where

$$\omega_1 = \sqrt{\omega^2 - \gamma^2}. \tag{15}$$

The constant A that can be chosen later for convenience. These complex coordinates are the natural variables (normal modes) of the system.

2.1 Complex Hamiltonian

We can think of the DSHO as a generalized hamiltonian system with a complex-valued hamiltonian.

The Poisson bracket $\{p, x\} = 1$ becomes, in terms of the variable z,

$$\{z^*, z\} = 2i\omega_1 |A|^2 \tag{16}$$

So if we choose $A = \frac{1}{\sqrt{2\omega_1}}$

$$\{z^*, z\} = i \tag{17}$$

So the **complex-valued** function

$$\mathcal{H} = (\omega_1 + i\gamma)zz^*. \tag{18}$$

satisfies

$$\frac{dz}{dt} = \{\mathcal{H}, z\}, \quad \frac{dz^*}{dt} = \{\mathcal{H}^*, z^*\} \tag{19}$$

Of course, the limit $\gamma \to 0$ this $\mathcal{H}$ tends to the usual hamiltonian $H = \omega zz^*$. Thus, on any analytic function ψ, we will have

$$\frac{d\psi}{dt} = \{\mathcal{H}, \psi\} = [\omega_1 + i\gamma]z\frac{\partial\psi}{\partial z} \tag{20}$$

2.2 Quantization

By the usual rules of canonical quantization, the quantum theory is given by turning $\mathcal{H}$ into a non-hermitean operator by replacing $z \mapsto a^\dagger$, $z^* \mapsto \hbar a$ and

$$[a, a^\dagger] = 1, \quad a^\dagger = z, \quad a = \frac{\partial}{\partial z}, \quad \mathcal{H} = \hbar(\omega_1 + i\gamma)a^\dagger a. \tag{21}$$

The effective hamiltonian $\mathcal{H} = H_1 + iH_2$ is normal (i.e., its hermitean and anti-hermitean parts commute, $[H_1, H_2] = 0$) so it is still meaningful to speak of eigenvectors of $\mathcal{H}$. The eigenvalues are complex

$$(\omega_1 + i\gamma)n, n = 0, 1, 2, \cdots. \tag{22}$$

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The higher excited states are more and more unstable. But the ground state is stable, as its eigenvalue is zero.

Thus a generic state

$$\psi = \sum_{n=0}^{\infty} \psi_n |n> \tag{23}$$

will evolve in time as

$$\psi(t) = \sum_{n=0}^{\infty} \psi_n e^{i\hbar[\omega+i\gamma]nt} |n> . \tag{24}$$

Unless ψ happens to be orthogonal to the ground state $|0>$, the wavefunction will tend to the ground state as time tends to infinity; final state will be the projection of the initial state to the ground state. This is the quantum analogue of the classical fact that the system will decay to the minimum energy state as time goes to infinity. All this sounds physically reasonable.

2.3   The Schrödinger Representation

In the Schrödinger representation,this amounts to

$$a = \frac{1}{\sqrt{2\hbar\omega_1}} \left[\omega_1 x + \hbar\frac{\partial}{\partial x} \right], \quad a^{\dagger} = \frac{1}{\sqrt{2\hbar\omega_1}} \left[\omega_1 x - \hbar\frac{\partial}{\partial x} \right] \tag{25}$$

$$\hat{\mathcal{H}} = \left(1 + i\frac{\gamma}{\omega_1}\right) \left[-\frac{\hbar^2}{2}\frac{\partial^2}{\partial x^2} + \tfrac{1}{2}\omega_1^2 x^2 - \tfrac{1}{2}\hbar\omega_1 \right] \tag{26}$$

Thus the operator representing momentum p is

$$\hat{p} = -i\hbar\frac{\partial}{\partial x} - \gamma x \tag{27}$$

which includes a subtle correction dependent on the friction.

The time evolution operator can be chosen to be

$$\hat{\mathcal{H}}_{\text{Schr}} = \hat{H} + \hat{H}_{\text{diss}} \tag{28}$$

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where

$$\hat{H} = -\frac{\hbar^2}{2}\frac{\partial^2}{\partial x^2} + \tfrac{1}{2}\omega^2 x^2 \tag{29}$$

is the usual harmonic oscillator hamiltonian and

$$\hat{H}_{\text{diss}} = -\tfrac{1}{2}\gamma^2 x^2 + i\frac{\gamma}{\omega_1}\left[-\frac{\hbar^2}{2}\frac{\partial^2}{\partial x^2} + \tfrac{1}{2}\omega_1^2 x^2 - \tfrac{1}{2}\hbar\omega_1\right] \tag{30}$$

This is slightly different from the operator $\hat{\mathcal{H}}$ above, because the ground state energy is not fixed to be zero. The constant in H_{diss} has been chosen so that this state has zero imaginary part for its eigenvalue.

3 Dissipative System of One Degree of Freedom

We will now generalize to a non-linear one-dimensional oscillator with friction:

$$\frac{dp}{dt} = -\frac{\partial V}{\partial x} - 2\gamma p, \quad \frac{dx}{dt} = p, \quad \gamma > 0. \tag{31}$$

The DSHO is the special case $V(x) = \tfrac{1}{2}\omega^2 x^2$. The idea is that we lose energy whenever the system is moving, at a rate proportional to its velocity. It again follows that

$$\frac{dH}{dt} = -2\gamma p^2 \leq 0 \tag{32}$$

where $H = \tfrac{1}{2}p^2 + V$. These equations can be written as

$$\frac{d\xi^i}{dt} = \{H, \xi^i\} - \gamma^{ij}\partial_j H \tag{33}$$

where $\gamma = \begin{pmatrix} 2\gamma & 0 \\ 0 & 0 \end{pmatrix}$ is a positive but degenerate matrix.

3.1 Complex Effective Hamiltonian

In the case of the DSHO, we saw that it is the combination

$$p_1 = p + \gamma x. \tag{34}$$

rather than p that appears naturally (for example in the normal co-ordinate z). In terms of (p_1, x) the equations above take the form

$$\frac{dp_1}{dt} = -\frac{\partial V_1}{\partial x} - \gamma p_1, \quad \frac{dx}{dt} = p_1 - \gamma x, \tag{35}$$

where

$$V_1(x) = V(x) - \tfrac{1}{2}\gamma^2 x^2; \tag{36}$$

i.e.,

$$\frac{d}{dt}\begin{pmatrix} p_1 \\ x \end{pmatrix} = \begin{pmatrix} \{H_1, p\} \\ \{H_1, x\} \end{pmatrix} - \gamma \begin{pmatrix} p_1 \\ x \end{pmatrix}. \tag{37}$$

We would like to see if we can write these as canonical equations of motion with a complex-valued effective hamiltonian. Suppose we define

$$H_2 = \frac{\gamma}{2\omega_2}[p^2 + \omega_2^2 x^2], \quad J = \begin{pmatrix} 0 & -\omega_2 \\ \frac{1}{\omega_2} & 0 \end{pmatrix} \tag{38}$$

for some constant ω_2 which we will choose later. Note that J is a complex structure; i.e.,

$$J^2 = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{39}$$

Then the equations of motion take the form

$$\frac{d}{dt}\begin{pmatrix} p_1 \\ x \end{pmatrix} = \begin{pmatrix} \{H_1, p_1\} \\ \{H_1, x\} \end{pmatrix} + J\begin{pmatrix} \{H_2, p_1\} \\ \{H_2, x\} \end{pmatrix}. \tag{40}$$

In terms of the complex co-ordinate

$$z = \frac{1}{\sqrt{2\omega_2}}[\omega_2 x - ip_1] \tag{41}$$

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this becomes

$$\frac{dz}{dt} = \{H_1 + iH_2, z\}. \tag{42}$$

Thus the non-linear oscillator also can be written as a canonical system with a complex valued hamiltonian $\mathcal{H} = H_1 + iH_2$; with $\{H_1, H_2\} \neq 0$ in general. But there are many ways to do this, parametrized by ω_2. The natural choice is

$$\omega_2 = \sqrt{V''(x_0) - \gamma^2}, \tag{43}$$

where x_0 is the equilibrium point at which $V'(x_0) = 0$. Then, in the neighborhood of the equilibrium point the complex structure reduces to that of the DSHO.

3.2 Quantization of a Dissipative One-dimensional System

We can quantize the above system by applying the usual rules of canonical quantization to the complex effective hamiltonian $H_1 + iH_2$ to get the operator:

$$\hat{\mathcal{H}} = -\frac{\hbar^2}{2}\frac{\partial^2}{\partial x^2} + V(x) - \tfrac{1}{2}\gamma^2 x^2 + i\frac{\gamma}{\omega_2}\left[-\frac{\hbar^2}{2}\frac{\partial^2}{\partial x^2} + \tfrac{1}{2}\omega_2^2 x^2 + c\right]. \tag{44}$$

The Schrödinger equation

$$-i\hbar\frac{\partial\psi}{\partial t} = \hat{\mathcal{H}}\psi \tag{45}$$

then determines the time evolution of the dissipative system.

The anti-hermitean part of the hamiltonian is bounded below so the imaginary part of the eigenvalues will be bounded below. We can choose the real constant c such that the eigenvalue with the smallest imaginary part is actually real. Then the generic state will evolve to this ' ground state'.

Remarks

1. Our model of dissipation amounts to adding a term

$$\hat{H}_{\text{diss}} = -\tfrac{1}{2}\gamma^2 x^2 + i\frac{\gamma}{\omega_2}\left[-\frac{\hbar^2}{2}\frac{\partial^2}{\partial x^2} + \tfrac{1}{2}\omega_2^2 x^2\right] + c \tag{46}$$

to the usual conservative hamiltonian

$$\hat{H} = -\frac{\hbar^2}{2}\frac{\partial^2}{\partial x^2} + V(x). \tag{47}$$

The dissipative term we add is very close to being anti-hermitean; i.e., except for the term $-\tfrac{1}{2}\gamma^2 x^2$ which is second order in the dissipation.

2. The Schrödinger equation for time reversed QM (evolving to the past rather than future) is $\mathcal{H}^\dagger$. The eigenvalues of $\mathcal{H}^\dagger$ are the complex conjugates of those of $\mathcal{H}$, but the eigenfunctions may not be the same as those of $\mathcal{H}$ since $[\mathcal{H}, \mathcal{H}^\dagger] \neq 0$ in general; i.e., the operator $\mathcal{H}$ may not be normal. (For the DSHO the commutator vanished so the issue did not arise.) Moreover the eigenfunctions of $\mathcal{H}$ corresponding to distinct eigenvalues need not be orthogonal; they would still be linearly independent, of course.

   There are examples of such non-normal hamiltonians in nature when time-reversal invariance is violated (e.g., $K\bar{K}$ oscillation)[10]. But in ordinary quantum mechanics such loss of time reversal invariance might be unsettling. Carl Bender [11] has suggested that for such non-hermitean hamiltonians, real eigenvalues and time reversal invariant dynamics can be recovered by modifying the inner product in the Hilbert space. This is the quantum counterpart to the reformulation of classical DSHO as a conservative canonical system by modifying the Poisson bracket and hamiltonian [4].

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However we note that in the presence of dissipation, the classical equations of motion are not time reversal invariant either: energy would grow rather than dissipate. So we should not expect quantum mechanics of dissipative systems to be time reversal invariant either. The appropriate symmetry is

$$[\mathcal{H}(\gamma)]^{\dagger} = \mathcal{H}(-\gamma) \tag{48}$$

which is satisfied in our case.

3. Note that within our model, the anti-hermitean part of the hamiltonian is a sort of harmonic oscillator even if the hermitean part has non-linear classical dynamics. This is because we chose a particularly simple form of dissipation, $\dot{p} \sim -\gamma p$. If we had chosen a more complicated (e.g., non-linear) form of dissipation, the anti-hermitean part would be more complicated. It is common to model a dissipative system by coupling it to a thermal bath of oscillators. The dissipation is determined by the spectral density of the frequencies of these oscillators. Each choice of spectral density leads to a different dissipation term and, in our description, to a different anti-hermitean part for the hamiltonian. But if the dissipative force is small it is reasonable to expect that it is linear in the velocity.

4 Comparison with the Approach of Caldeira and Leggett

It is interesting to see how our approach compares with the influential work of Caldeira and Leggett [8]

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Their basic idea is to couple the one dimensional system to a collection of harmonic oscillators; the transfer of energy to the oscillators leads to its dissipation in the original system. The lagrangian is[1]

$$L = \tfrac{1}{2}\dot{x}^2 - V(x) + \sum_\alpha [\tfrac{1}{2}m_\alpha \dot{q}_\alpha^2 - \tfrac{1}{2}m_\alpha \omega_\alpha^2 q_\alpha^2] - x \sum_\alpha C_\alpha q_\alpha - \tfrac{1}{2}\Delta\omega^2 x^2. \quad (49)$$

In the last term, $\Delta\omega^2$ is chosen such that the equilibrium point of the variable x is still the minimum of V: it is a finite renormalization of the potential induced by the coupling to the oscillators.

In the euclidean path integral approach, they derive an effective action after integrating out these oscillators:

$$S_{\text{eff}}[x] = \int_0^T [\tfrac{1}{2}\dot{x}^2 + V(x)]d\tau + \tfrac{1}{2}\int_{-\infty}^\infty d\tau' \int_0^T d\tau [x(\tau) - x(\tau')]^2 a(\tau - \tau') \quad (50)$$

where

$$a(\tau) = \sum_\alpha \frac{C_\alpha^2}{4m_\alpha \omega_\alpha} e^{-\omega_\alpha \tau} \equiv \frac{1}{2\pi} \int_0^\infty \rho(\omega) e^{-\omega\tau} d\omega \quad (51)$$

All the properties of the oscillators that matter to us are contained in the function $\rho(\omega)$. The choice $\rho(\omega) = \gamma\omega$ corresponds to the usual dissipative term . This at first sounds strange: how can this effective action that is non-local in time be related to the differential equation of the damped oscillator? The minimum of S_{eff} (which determines the tunneling amplitude in the leading WKB approximation) is given by the integro-differential equation[2]

$$\frac{d^2 x}{d\tau^2} = \frac{\partial V}{\partial x} + \frac{\gamma}{\pi} \mathcal{P} \int_{-\infty}^\infty d\tau' \frac{x(\tau) - x(\tau')}{(\tau - \tau')^2}. \quad (52)$$

In the absence of damping we would get this equation for the most likely path in the classically forbidden region by the replacement [12] :

$$t \mapsto i\tau \quad (53)$$

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[1] We change notation slightly from them to avoid conflict with ours.

[2] Here, $\mathcal{P}$ stands for the Cauchy Principal part.

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in Newton's law:

$$\frac{d^2x}{dt^2} = -\frac{\partial V}{\partial x}.$$ (54)

Such a naive replacement would not make sense in the damped equation:

$$\frac{d^2x}{dt^2} = -\frac{\partial V}{\partial x} - \gamma \frac{dx}{dt} \mapsto \frac{d^2x}{d\tau^2} = \frac{\partial V}{\partial x} - i\gamma \frac{dx}{d\tau}.$$ (55)

Since x is real, the multiplication by i on the r.h.s. does not make sense.

Perhaps multiplication by i should be replaced by the action of some linear operator on the space of functions of one variable? If so the square of this linear operator must be -1. There is a well-known operator with this property, the Hilbert transform [13]:

$$\mathcal{J}x(t) = \frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty} \frac{x(t')}{t - t'}dt'$$ (56)

The Caldeira-Legget equation of motion is obtained by replacing the i in the naive equation above by the operator $\mathcal{J}$:

$$\frac{d^2x}{d\tau^2} = \frac{\partial V}{\partial x} - \gamma \frac{d}{d\tau}\mathcal{J}x.$$ (57)

Thus although obtained by very different arguments, there is a way to understand the Caldeira-Legget equation of motion in terms of a complex structure (the Hilbert transform) on the space of paths. We instead introduce a complex structure in the phase space at the classical level. Could ours be the canonical quantization of their non-local path integral?

5   Tunneling in a Dissipative System

Perhaps the most interesting question about quantum dissipative systems is how dissipation affects tunneling. The standard WKB approximation method adapts easily to our case.

For illustrative purposes, it suffices to consider a one-dimensional system with a potential

$$V(x) = \tfrac{1}{2}\omega^2 x^2 \left(1 - \frac{x}{a}.\right) \tag{58}$$

We ask for the tunneling probability amplitude from the origin $x = 0$ to the point $x = a$ in a long time. In the absence of dissipation this is given in the WKB approximation by

$$e^{-\frac{1}{\hbar}\int_0^a \sqrt{2V(x)}dx}. \tag{59}$$

The integral in the exponent is the minimum of the imaginary time action

$$\int_0^\infty [\tfrac{1}{2}\dot{x}^2 + V(x(\tau))]d\tau \tag{60}$$

among all paths satisfying the boundary conditions $x(0) = 0$ and $x(\infty) = a$. (This minimizing path is the 'instanton'.)

If we apply the WKB approximation to the Schrödinger equation we get

$$\psi = e^{-\frac{\phi}{\hbar}}, \quad -\tfrac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 + V_1(x) + i\frac{\gamma}{\omega_2}\left[-\tfrac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 + \tfrac{1}{2}\omega_2^2 x\right] \approx 0. \tag{61}$$

Solving this,

$$\phi(x) = \int_0^x \sqrt{\frac{2V_1(x) + i\gamma\omega_2 x^2}{1 + i\frac{\gamma}{\omega_2}}}dx \tag{62}$$

The tunneling probability is given by

$$e^{-2\mathrm{Re}\ \phi(b)} \tag{63}$$

where b is the point of escape from the potential barrier; it might depend on the dissipation.

It looks most natural to choose $\omega_2 = \omega_1 = \sqrt{\omega^2 - \gamma^2}$ as in the case of the DSHO. Then,

$$\phi(x) = \omega_1 \int_0^x \sqrt{1 - \frac{\omega^2}{\omega_1(\omega_1 + i\gamma)}\frac{x}{a}}\ x dx. \tag{64}$$

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Now,

$$\int_0^b x\sqrt{b-x}\,dx = \frac{4}{15}b^{\frac{5}{2}}.$$

(65)

There are small discrepancies (up to higher order terms in γ) depending on whether we think of the point of escape as a, or $\frac{\omega_1^2}{\omega^2}a$ or the complex number $\frac{\omega_1(\omega_1+i\gamma)}{\omega^2}a$ which is the zero of the integrand. The last choice gives the simplest answer for the tunneling probability

$$e^{-\frac{8}{15}\frac{a^2\omega^2}{\hbar\omega}\frac{(\omega^2-\gamma^2)^{\frac{3}{2}}(\omega^2-2\gamma^2)}{\omega^5}}$$

(66)

This differs from the results of Caldeira and Legget: the tunneling probability is enhanced by dissipation. There could still be systems in nature that are described by our model.

6 Multi-Dimensional Dissipative Systems

Now we will generalize to a multi-dimensional dynamical system while also allowing for a certain kind of non-linearity in the frictional force.

Suppose the hamiltonian is the sum of a kinetic and a potential energy,

$$H = \tfrac{1}{2}p_a p_a + V(x).$$

(67)

With the usual Poisson brackets

$$\{p_a, x^b\} = \delta_a^b, \quad \{p_a, p_b\} = 0 = \{x^a, x^b\}$$

(68)

the conservative equations of motion are

$$\frac{dp_a}{dt} = -\partial_a V, \quad \frac{dx^a}{dt} = p^a$$

(69)

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We now assume that the frictional force is of the form $-2\gamma_{ab}\dot{x}^b$ for some positive non-degenerate tensor γ_{ab} that might depend on x. The idea is that the system loses energy when parts of it move relative to other parts or relative to some medium in which the system is immersed. Then the equations of motion become

$$\frac{dp_a}{dt} = -\partial_a V - 2\gamma_{ab}p_b, \quad \frac{dx^a}{dt} = p^a. \tag{70}$$

Here we raise and lower indices using the flat euclidean metric δ_{ab}. So

$$\frac{dH}{dt} = -2\gamma^{ab}p_a p_b \leq 0. \tag{71}$$

If the dissipation tensor happens to be the Hessian of some convex function

$$\gamma_{ab} = \partial_a \partial_b W, \tag{72}$$

it is possible to write these as Hamilton's equations with a complex-valued hamiltonian.

Then, we would have

$$\frac{d}{dt}\partial_a W = \gamma_{ab}p^b, \quad \gamma_{ab}\partial_b W = \partial_a(\tfrac{1}{2}\partial_b W \partial_b W). \tag{73}$$

Using these identities we can rewrite the equations of motion in the new variables:

$$\tilde{p}_a = p_a + \partial_a W \tag{74}$$

as

$$\frac{d\tilde{p}_a}{dt} = -\partial_a[V - \tfrac{1}{2}(\partial W)^2] - \gamma_{ab}\tilde{p}_b, \quad \frac{dx_a}{dt} = \tilde{p}_a - \partial_a W. \tag{75}$$

This motivates us to define

$$H_1 = \tfrac{1}{2}\tilde{p}^2 + V - \tfrac{1}{2}(\partial W)^2, \quad H_2 = \frac{1}{\omega_2}[\tfrac{1}{2}\gamma_{ab}\tilde{p}_a\tilde{p}_b + \omega_2^2 W], \quad J = \begin{pmatrix} 0 & -\omega_2 \\ \frac{1}{\omega_2} & 0 \end{pmatrix} \tag{76}$$

for some positive constant ω_2. Note that J is a complex structure, $J^2 = -1$. Then

$$\frac{d}{dt}\begin{pmatrix} \tilde{p} \\ x \end{pmatrix} = \begin{pmatrix} \{H_1, \tilde{p}\} \\ \{H_1, x\} \end{pmatrix} + J \begin{pmatrix} \{H_2, \tilde{p}\} \\ \{H_2, x\} \end{pmatrix} \tag{77}$$

In terms of the complex variable

$$z_a = \frac{1}{\sqrt{2\omega_2}}[\omega_2 x_a - i\tilde{p}_a] \tag{78}$$

this is just

$$\frac{dz_a}{dt} = \{H_1 + iH_2, z_a\} \tag{79}$$

just as before. Thus once the dissipation tensor is of the form $\gamma_{ab} = \partial_a \partial_b W$ the whole framework generalizes easily. The effect of dissipation is to add the term

$$H_{\mathrm{diss}} = -\tfrac{1}{2}(\partial W)^2 + \frac{i}{\omega_2}[\tfrac{1}{2}\gamma_{ab}p_a p_b + \omega_2^2 W] \tag{80}$$

to the hamiltonian. The classical equations turn out to be independent of the choice of ω_2, but the quantum theory will depend on its choice.

6.1 Quantization

We can then quantize this operator in the Schrödinger picture:

$$\hat{\mathcal{H}} = \hat{H} + \hat{H}_{\mathrm{diss}}, \tag{81}$$

where

$$\hat{H}\psi = -\tfrac{1}{2}\nabla^2\psi + V\psi, \quad H_{\mathrm{diss}}\psi = -\tfrac{1}{2}(\nabla W)^2\psi + \frac{i}{\omega_2}\left[-\frac{1}{2}\partial_a\left(\gamma^{ab}\partial_b\psi\right) + \omega_2^2 W\right] \tag{82}$$

A technical point to note here is that there are two competing metrics in the story. The kinetic energy is $\tfrac{1}{2}p_a p_a$, determined by the euclidean metric. But the

quadratic term in the dissipative part of the hamiltonian is determined by some other tensor γ_{ab}. We have chosen to use the Euclidean metric δ_{ab} to define derivatives and to raise and lower indices. Thus, $\gamma^{ab} = \delta^{ac}\delta^{bd}\gamma_{cd}$ and not the inverse of γ_{ab}.

The operator $\partial_a \left(\gamma^{ab}\partial_b\psi \right)$ is thus a kind of 'mixed' laplacian that uses γ_{ab} as well as, implictly, the euclidean metric. Since $\partial_a\gamma^{bc} \neq 0$ in general, the ordering of factors is important. with the order we chose,$\partial_a \left(\gamma^{ab}\partial_b\psi \right)$ is hermitean and positive.

7 Geometric Model of Dissipative Mechanics

We will now further generalize to the case of dissipative dynamics on a cotangent bundle T^*Q.

The hamiltonian is again the sum of kinetic and potential energies

$$H = \tfrac{1}{2}g^{ab}p_ap_b + V(x). \tag{83}$$

except that we now allow the tensor g^{ab} not to be constant. We will use g_{ab} (the inverse of g^{ab}) as the Riemann metric, used to define covariant derivative ∇_a and to raise and lower indices.

The conservative equations of motion are

$$\frac{dp^a}{dt} + \Gamma^a_{bc}p^bp^c = -g^{ab}\partial_bV, \quad \frac{dx^a}{dt} = p^a \tag{84}$$

where Γ^a_{bc} is the usual Christoffel symbol.

With friction added,

$$\frac{dp^a}{dt} + \Gamma^a_{bc}p_bp^c = -g^{ab}\partial_bV - 2\gamma^{ab}p_b, \quad \frac{dx^a}{dt} = p^a \tag{85}$$

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where γ^{ab} is the dissipation tensor, assumed to be positive and non-degenerate. Again, if the dissipation tensor is the Hessian of a convex function

$$\gamma_{ab} \equiv g_{ac}g_{bd}\gamma^{bd} = \nabla_a\partial_b W, \tag{86}$$

there are simplifications because

$$\frac{d}{dt}(\nabla^a W) + \Gamma^a_{bc}\nabla^b W p^c = \gamma_{ab}p^b \tag{87}$$

We define again

$$\tilde{p}_a = p_a + \partial_a W \tag{88}$$

to get

$$\frac{d}{dt}\begin{pmatrix} \tilde{p} \\ x \end{pmatrix} = \begin{pmatrix} \{H_1, \tilde{p}\} \\ \{H_1, x\} \end{pmatrix} + J\begin{pmatrix} \{H_2, \tilde{p}\} \\ \{H_2, x\} \end{pmatrix} \tag{89}$$

where

$$H_1 = \tfrac{1}{2}g^{ab}\tilde{p}_a\tilde{p}_b + V - (\nabla W)^2, \quad H_2 = \frac{1}{2\omega_2}\gamma^{ab}\tilde{p}_a\tilde{p}_b + \omega_2 W, \quad J = \begin{pmatrix} 0 & -\omega_2 \\ \frac{1}{\omega_2} & 0 \end{pmatrix}. \tag{90}$$

Since again $J^2 = -$ it is an almost complex structure; but it may not be integrable in general. Every tangent bundle has a natural almost complex structure; J is simply its translation to the co-tangent bundle T^*Q using the metric g_{ab} which identifies the tangent and co-tangent bundles.

Because J may not be integrable, we are not able to rewrite this in terms of a complex co-ordinate z in general. Nevertheless, we can think of the above equations as a generalization of hamilton's equations to a complex hamiltonian $H_1 + iH_2$.

Clearly, it is possible to quantize these system by applying the correspondence principle. Since the ideas are not very different, we will not work out the details. The hamiltonian is:

$$\hat{\mathcal{H}} = \hat{H} + \hat{H}_{\text{diss}}, \tag{91}$$

where

$$\hat{H}\psi = -\tfrac{1}{2}\nabla^2\psi + V\psi, \quad H_{\text{diss}}\psi = -\tfrac{1}{2}g^{ab}\partial_a W \partial_b W \psi + \frac{i}{\omega_2}\left[-\frac{1}{2}\nabla_a\left(\gamma^{ab}\partial_b\psi\right) + \omega_2^2 W\right] + ic. \tag{92}$$

The imaginary constant ic is chosen such that the ground state is stable.

8 Acknowledgement

References

[1] U. Weiss, *Quantum Dissipative Systems* , World Scientific, Singapore (1999)

[2] W. Schleich, *Quantum Optics in Phase Space* Wiley (2001).

[3] M. O. Scully and M. S. Zubairy,*Quantum Optics* Cambridge Univ. Press (1997).

[4] S. Okubo Phys. Rev. A 23 (1981) 2776; G. Bimonte, G. Esposito, G. Marmo and C. Stornaiolo Phys. Lett. A 318(2003) 313.

[5] E. Kanai, Prog. Theor. Phys. 3(1948) 440 .

[6] P. J. Morrison Physica D 18(1986) 410; A. N. Kaufman Phys. Lett. A 100 (1984) 419.

[7] R. Krechetnikov and J. E. Marsden, *Dissipation Induced Instabilities in Finite Dimensions* to appear in Rev. Mod. Phys.;N. M. Bou-Rabee, J. E. Marsden, and L. A. Romero, SIAM J. Applied Dynamical Systems, 3 (2004), 352; A. Bloch, P. S. Krishnaprasad, J.E. Marsden and T. S. Ratiu, Comm. Math. Phys. 175 (1996)1.

[8] A. O. Caldeira and A. J. Leggett Ann. Phys. 149 (1983) 374; Physica A121(1983) 587.

[9] N. Hatano and D. R. Nelson Phys. Rev. Lett. 77 (1996) 570; D. R. Nelson and N. M. Shnerb Phys. Rev. E 58 (1998) 1383.

[10] V. S. Mathur and S. G. Rajeev Mod. Phys. Lett. **A6**, 2741 (1991).

[11] C. M. Bender and S. Boettcher, Phys. Rev. Lett. 80(1998) 5243.

[12] C. Callan and S. Coleman Phys.Rev. D16 (1977) 1762.

[13] E. T. Copson, *Theory of Functions of a Complex Variable*, Oxford University Press, London(1935); E. C. Titchmarsh, *Introduction to the Theory of Fourier Integrals* Oxford, Clarendon Press (1948).